GRAPHICAL THERMODYNAMICS OF THE FREE AIR

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In any attempt to increase our understanding of weather phenomena, the investigation of energy relations in the earth's atmosphere is necessarily of fundamental importance; and one of the needs of modern meteorology is that of a method for the observation and representation of the physical characteristics of the free air in such a way as to make the scientific application of thermodynamics to meteorological work a practical possibility. The national meteorological institutions of the world have been established, and are maintained, at the public expense primarily for the sake of the practical services which they can render; these services must be given without fail, day after day, to the best of our ability, and take precedence over everything else, but there should also be a constant effort made to improve and extend them in order to meet still more satisfactorily the increasing demands of the public. Marked and fundamental improvements in meteorological practice, however, probably can come, if at all, only as a result of effecting and utilizing further advances in the pure science of meteorology; empirical methods, while of very great value, and in the present state of our knowledge indispensable, have serious limitations, which are soon reached, and continued investigation along empirical or semi-empirical lines, while also useful and necessary at present, can result in only limited further improvement. However inevitable empirical methods may be by reason of the difficulty of finding exact and complete solutions of the complex problems of the atmosphere, they constitute but one stage in the historical development of meteorology and its applications, and steady progress away from them is taking place. Particularly encouraging are the advances which have been made in thermodynamical meteorology.

The investigation of meteorological problems involves three steps: (1) The extension of our knowledge of the theoretical physics of the atmosphere; (2) the discovery of how to use this knowledge to attain desired accomplishments; and (3) the invention of means for making this use practicable under the conditions of daily meteorological work. All the physical processes of the atmosphere have turned out to be even more intricate than the earlier meteorologists thought them to be, and progress along each of the above three lines is necessarily slow and difficult; nevertheless every possible opportunity should be provided for further research (*1*).

The general aim of thermodynamics is the investigation of the states or conditions of material systems, as defined by the properties or qualities of the systems, and of the manner in which the variable qualities change relatively to one another as the systems undergo changes of state while receiving or giving out various forms of energy. The actually existing physical state of the atmosphere at any given time and place may be determined by aerological soundings; and the methods of thermodynamics provide means of utilizing the data so obtained in the scientific investigation of the changes of condition that constitute the sequence of weather. Great difficulties are introduced, however, by the presence in the atmosphere of highly variable quantities of water vapor; and only after many years of study has success been achieved in finding rapid and convenient, yet accurate, methods for the application of thermodynamical laws to meteorological phenomena.

FUNDAMENTAL THERMODYNAMICAL RELATIONS FOR ATMOSPHERIC AIR

The characteristic equation of state for dry air,

$$pv = MRT,\qquad p = \rho RT, \tag{1}$$

follows directly from experiment, and together with the First Law of Thermodynamics (conservation of energy) leads to the energy equation

$$dQ = c_v dT + Apdv = c_p dT - Avdp,\ \text{per unit mass;} \tag{2}$$

from (1) and (2) may be obtained the isentropic equation for adiabatic processes,

$$pv^\gamma = const. = f(\varphi), \tag{3}$$

and Poisson's Equation

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}},\ or\ \frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{\frac{\gamma}{\gamma-1}} \equiv \left(\frac{T}{T_0}\right)^m, \tag{4}$$

which gives the relation between temperature changes and pressure changes in adiabatic processes. In the above equations, p denotes gas pressure; M, mass; ρ, density; R, the gas constant (2.870 x 10^6 c. g. s. for dry air, referred to 1000 mb. and 273° Abs.); T, absolute temperature; Q, energy in thermal units; c_v, specific heat of dry air at constant volume (0.1715); A, the reciprocal of the mechanical equivalent of heat [$1/(4.18\text{x}10^7)$]; v, volume; c_p, specific heat of dry air at constant pressure (0.2417); φ, entropy; γ, ratio of the specific heats of dry air (1.40); $m = 1/0.286$; and the zero subscripts refer to (arbitrary) initial conditions.

If the air contains water vapor, the preceding equations still hold very closely as long as saturation is not attained, only the third decimal place in the value of $(\gamma-1)/\gamma$ being changed (0.288 is a fair average value); if great accuracy is desired, however, we must replace (1) by

$$p = R\left(\rho + \frac{\rho''}{\epsilon}\right)T, \tag{5}$$

where p is the pressure of the mixture of dry air and water vapor, ρ the density of the air in the mixture, ρ'' the density of the water vapor constituent; $\rho' = \rho + \rho''$, ρ' being the density of the mixture; and ϵ is the specific gravity of aqueous vapor (0.622). If the specific humidity, or weight of water vapor per unit weight of the mixture, be denoted by $q = \rho''/\rho'$, then

$$p = \rho' R(1 + 0.605q)T \equiv \rho' R'' T \equiv \rho' RT'', \tag{6}$$

where T'' is the virtual temperature; or, again, if the "mixing ratio," or weight of water vapor per unit weight of dry air present in the mixture, be $x = \rho''/\rho$, we have

$$p = \rho R\left(1 + \frac{x}{\epsilon}\right)T \equiv \rho R' T; \tag{7}$$

x is the weight of aqueous vapor in a weight $(1+x)$ of the mixture, and $q = x/(1+x)$, while $x = q/(1-q)$.

For moist air, the energy equation (2) becomes, for weight $(1+x)$ of mixture,

$$dQ = (c_p + xc''_p)dT - A\left(1 + \frac{x}{\epsilon}\right)RT\frac{dp}{p}, \tag{8}$$

whence m in (4) has the value

$$m=\frac{c_p}{AR}\left[\frac{1+\frac{c''_p}{c_p}x}{1+\frac{x}{\epsilon}}\right]=3.441\left(\frac{1+2.023x}{1+1.608x}\right), \quad (9)$$

in which c''_p is the specific heat of unsaturated water vapor at constant pressure.

In meteorology we are usually interested in processes involving continuously decreasing pressure (as in upward convection). The preceding equations supply a sufficient description of such processes as long as saturation is not attained, i. e., throughout the "dry stage"; when condensation begins, these equations cease to apply, and the "rain," "hail," and "snow" stages are characterized by more complicated equations, such as

$$\log\frac{p'}{p'_0}=\frac{c_p+\xi c}{AR}\log\frac{T}{T_0}+\frac{M}{AR}\left(\frac{xr}{T}-\frac{x_0r_0}{T_0}\right), \quad (10)$$

which describes the rain stage; in this equation ξ is the total weight of water present (liquid and vapor) per unit weight of dry air present, c the specific heat of water, r the latent heat of evaporation, p' the partial pressure of the dry air, M the logarithmic modulus, and the subscripts refer to initial conditions.

In the applications of thermodynamics to practical meteorological work, the hail and snow stages may be ignored, as Shaw has pointed out.

When rain begins to fall, there is a decrease in the total energy of the air; the condition in which there is no addition or subtraction of heat, but in which all condensed water falls out, is called "pseudo-adiabatic," or irreversible adiabatic, in contradistinction to the true adiabatic or "reversible adiabatic" condition; the above adiabatic equations can be reduced to the pseudo-adiabatic by dropping the water terms.

Of fundamental importance in thermodynamical meteorology is the quantity known as potential temperature: The potential temperature of a mass of air is the absolute temperature that would result if the air were brought adiabatically to some arbitrary standard pressure which Shaw selects to be 1,000 mb. (Shaw names the potential temperature referred to this standard the "megatemperature.") The potential temperature derives its importance from its close connection with the basic (but abstract and elusive) thermodynamical concept, *entropy;* from the characteristic equation and the equation of energy we have

$$\phi=c_p\log\frac{T}{T_0}-AR\log\frac{p}{p_0}\text{ per unit mass,} \quad (11)$$

in which the zero subscripts refer to the (arbitrary) zero point from which the entropy, ϕ, of the *dry air* is reckoned, and which Shaw takes to be 200° Abs. and 1,000 mb. In terms of potential temperature, θ, we have by (11) and (4),

$$\phi=\log_e\left(\frac{\theta}{200}\right)^{c_p}=c_p\log\theta+const.\ \frac{\text{joules}}{\text{degree}}. \quad (12)$$

The total entropy of *moist air* includes the entropy, ϕ, of the dry air, and also that of the water vapor; the potential temperature is merely another form of expression for the entropy of the dry air, no allowance being made for the latent heat of the water vapor mixed with this air; the quantity ϕ above may be referred to as "realized entropy," having been realized in the form of temperature. The meteorologist may merely look upon ϕ as a numerical magnitude, *defined* by (12), proportional to the potential temperature. The vertical distribution of potential temperature determines the stability of the atmosphere, and the possibility and extent of convectional processes. Potential temperature may be increased by heat of condensation, but the total entropy of the mixture remains unchanged.

Numerous tables have been prepared and published to facilitate computations with the above equations; and from these tables, graphs may be prepared if desired.

GRAPHICAL REPRESENTATION OF THE THERMODYNAMICAL STATE OF THE FREE ATMOSPHERE

The common method of portraying the physical state of the atmosphere by plotting pressure, temperature, etc., against height is entirely unsuited to the use of the data for the scientific investigation of the thermodynamical mechanism of meteorological phenomena; it is necessary to plot the data in some one of the forms employed in thermodynamics. The method first introduced into meteorology is that embodied in the well-known Neuhoff diagram (*2*).

The Neuhoff diagram shows what will happen during the ascent of air which contains a given amount of water vapor. This diagram consists of a groundwork of adiabatic lines for dry air and reversible adiabatics for saturated air, computed from equations such as (4) and (11), referred to temperature and logarithm of pressure as coordinates. It is also possible to graph the irreversible adiabatics. In reality, during ascent (due, e. g., to local heating above the temperature of the environment) the irreversible adiabatics are the ones approximately followed, while on descent the lines followed will be practically indistinguishable from those for dry air. On the groundwork may be superposed the graph of the data obtained from an aerological sounding; the resulting diagram is highly useful in the study of thermodynamical processes in the atmosphere, e. g., thermal convection (which has been found to present some of the most difficult problems of meteorology and to be not at all the simple and well understood phenomenon it formerly was thought to be), as well as in the graphical reduction and representation of daily upper air observations, and their application to the determination of the structure of the atmosphere.

The Neuhoff diagram has been adapted to practical use in daily meteorological work at the Lindenberg Observatory (*3*). The conception of great masses of moving air, differing to a greater or less extent from one another in respect to temperature, humidity, velocity, etc., and separated by more or less well-marked "surfaces," plays an important rôle in modern meteorological ideas of the dynamic mechanism of atmospheric phenomena; it is, e. g., a leading element among the ideas of the Bjerknes school, now being successfully applied to forecasting in several European countries. The location of the air streams and boundary surfaces is much facilitated by aerological observations, frequently being difficult or impossible from surface observations alone; but for this purpose the data must be put in proper form, and, for practical use such as in forecasting, methods must be available for doing this very quickly.

The Lindenberg Observatory employs what is called *adiabatic paper*, upon which are printed the dry air adiabatics (computed from equation (4), with $1/m=0.288$ and $p_0=750$ mm. or 1,000 mb.) referred to temperature and logarithm of pressure as rectangular coordinates, and curves of specific humidity computed from

$$q=\epsilon\frac{e}{b-e\ (1-\epsilon)}, \quad (13)$$

in which e is the vapor pressure and b the barometric pressure. Temperature, vapor pressure, relative humidity, and wind are each plotted against pressure, the fundamental meteorological variable; printed scales are provided for the first two, while the other two are entered at any convenient place and the scales written in (using, e. g., 10° of the temperature scale for the range 0% to 100% of humidity). Provision is made for obtaining virtual temperatures graphically by Bjerknes' method. *See Fig. 1*, which is fully explained in the legend.

A pressure-height curve, with heights as abscissae, may be drawn at any convenient place on the diagram; and may be constructed either from heights computed in the ordinary way, or graphically; for convenience, an altitude scale may then be written alongside the pressure scale of ordinates also.

The pressure-height curve is constructed graphically out of a number of parts or sections, each of which is drawn parallel to the adiabatic which falls nearest the mean temperature of a stratum that has a uniform lapse rate; such strata are indicated in Figure 1 by the portions of the temperature curve between the heavy horizontal lines. It is obvious that where no marked change in the lapse rate occurs, the mean temperature of the stratum will be the same no matter what the lapse rate may be, provided the lapse rate curve be drawn through the midpoint of the stratum, i. e., the point whose actual temperature is equal to the mean temperature of the stratum. The nearer the actual lapse is to the adiabatic lapse rate, the greater the pressure interval may be taken to be, i. e., the fewer the sections making up the complete pressure-height curve. In the case of pronounced temperature inversions, therefore, care must be taken to use sufficiently small pressure intervals. In the construction of the curve by the above method, it is assumed that the adiabatic lapse rate is 1°/100 m., hence the resulting indicated heights must be increased by 1.6 per cent if accuracy is desired.

Potential temperatures and specific humidities may be read off immediately from the diagram; the former are obtained from the intersections of the pressure-temperature curve with the adiabatics, the latter from the intersections of the pressure-vapor pressure curve with the specific humidity curves. Potential temperatures have been very little used in practical meteorology, probably because of the time required for their calculation; however, under many circumstances, e. g., when large pressure differences exist in time or space, comparisons based on actual temperatures may be quite misleading, potential temperatures being preferable.

In Figure 1 and Table I are shown the results, obtained graphically, of a kite flight made at Ellendale, N. Dak., on February 2, 1925, chosen because of the extreme deviation of the lapse rate from the dry air adiabatic. It will be seen that the differences between the altitudes determined graphically and those found by computation are all small, the former in practically every case being less than 1 per cent in error as compared to the latter. Such a graph gives the complete data of the flight in very compact and readily comprehensible form; values of the meteorological elements may be read off for any desired level as well as for any given pressure, and the corresponding altitude for any pressure is directly obtained; pronounced temperature inversions, and all other interesting features present are evident at a glance; and the accuracy is nearly as great as that of the customary tedious computation, with its resulting tabulation that fails to exhibit the conditions in any conspicuous manner or to furnish potential temperatures, etc.

When great accuracy is desired, the use of the graphical method affords but little gain in time; but for immediate practical use in forecasting, a sufficiently accurate graph can be constructed within a few minutes, directly from the data as recorded on the meteorogram. In Figure 1, the temperature curve is divided up into a relatively large number of subdivisions, each having a different lapse rate; it should be clearly understood that for use in forecasting it would be sufficiently accurate to divide this curve into only two divisions, and thus to construct the pressure-height curve in a negligible amount of time.

TABLE I

[Ellendale, N. Dak., Feb. 2, 1925]

Altitude by computation	Altitude by graph	Difference
(m.) m. s. l.	(m.) m. s. l.	(m.) m. s. l.
533	549	+16
1362	1361	−1
1497	1494	−3
1917	1900	−17
2783	2763	−20
2930	2926	−4
3286	3292	+6
3669	3658	−11
3800	3780	−20
4240	4211	−29
4316	4308	−8
4358	4328	−30

However, the use of temperature and pressure (or logarithm of pressure) as coordinates has the distinct disadvantage that energy can not be represented graphically; and energy relations are what we are ultimately interested in when investigating the physical phenomena of the atmosphere. On the common pressure-specific volume "indicator diagram," energy in dynamical units is represented by areas, but the areas enclosed between adiabatics and isothermals are greatly elongated parallelograms, so that interpretation and application are difficult. After a prolonged investigation of various forms of thermodynamical diagrams, Sir Napier Shaw and his colleagues have finally selected temperature and entropy as the most suitable coordinates, and have set out the results of aerological soundings by means of what they call the *tephigram* (tē-phĭ-gram) (*4*).

THE TEPHIGRAM

The groundwork of the tephigram is essentially the Neuhoff diagram transformed to temperature-entropy coordinates, and extended to lower pressures and temperatures; for the convenience of the meteorologist, however, the coordinates are represented as temperature and logarithm of potential temperature. This groundwork enables one to follow the changes which will take place in the condition of dry air, or in the condition of air originally saturated, as the pressure is reduced in any adiabatic process; dry air alone is regarded as the "working substance," or substance that goes through the thermodynamic changes, and any moisture carried along is regarded as a reservoir of latent energy—a possible supply of heat—that becomes realized when condensation takes place, all other effects of water vapor being neglected. Only the realized entropy is shown; loss of water, and latent heat of condensation, are allowed for by increasing the realized entropy of the dry air accordingly. The dry and the irreversible saturation adiabatics, isobaric lines, and lines showing the number of grams of water vapor necessary to saturate one kilogram of dry air are shown. Areas represent energy.

The groundwork shows the environment of pressure through which dry or saturated air would have to pass if it ascended either spontaneously or through being forced up; and to bring the condition of the air in a vertical section of the atmosphere at any time, as revealed by an aerological sounding, into relation with the thermodynamical properties of dry and saturated air as shown by the temperature-entropy diagram, the sounding is plotted on the groundwork; the resulting representation is called a tephigram. The tephigram affords facilities not hitherto available for studying the physical processes in the atmosphere; and many practical applications are possible. See Figure 2, fully explained in the legend.

Spontaneous adiabatic ascent of unsaturated air, in which no kinetic energy or momentum is shared with the environment, will be along a horizontal isentropic line on the diagram (since the potential temperature remains constant); spontaneous ascent of saturated air will be along the saturation adiabatic through the starting point; in any case, a return journey will be along a horizontal line. The amount of water condensed out can be determined from the diagram. Spontaneous ascent of dry air can therefore occur only in regions where the graph of the sounding slopes downward from a horizontal line, implying instability for dry air, i. e., a superadiabatic lapse rate; stability is indicated by a deviation of the graph upward from the horizontal, regions of convective equilibrium by no deviation from a horizontal (constant potential temperature), inversions by a deviation to the left of the vertical, isothermal regions by no deviation from a vertical. Spontaneous ascent of saturated air can occur only in regions where the saturation adiabatic through the starting point keeps on the warm side of, i. e., above, the tephigram, implying instability for saturated air; a deviation of the graph of the sounding upward from the saturation adiabatic indicates stability for saturated air. Spontaneous ascent of dry air will continue until the tephigram again intersects the horizontal isentropic through the starting point, when the rising air will come to equilibrium, possibly after some oscillations (unless in the course of its ascent it has become saturated, in which case a new set of conditions will supervene); spontaneous ascent of saturated air will continue, perhaps with increased acceleration (condensed water being lost on the way), until the tephigram again intersects the saturation adiabatic through the starting point, when the ascending air will come to rest after oscillations of more or less violence or the transformation of its potential energy into some stable kinetic form, the final potential temperature being marked by the ordinate of the extreme point reached.

The energy consumed or given out per kilogram of (dry or saturated) air in any convectional process is represented by the area inclosed by the tephigram and the adiabatic followed during the process between the two points of intersection (starting point and point of equilibrium); in the case of spontaneous ascent the area is positive, the tephigram lying below the adiabatic, and when spontaneous ascent is not possible the area is negative, lying below the tephigram; a positive area is to be described by going around it in a clockwise direction.

Similarly, a complete thermodynamic cycle can easily be followed out on the tephigram groundwork, and the efficiency determined.

The groundwork of the tephigram refers only to saturated air, but the air represented by any point on the graph of the sounding may be in any state of humidity. A dew-point curve, or *depegram* (dē-pē-gram), may also be plotted, however; the isobaric curve through a point of the tephigram is followed until it intersects the temperature line corresponding to the dew point, and the intersection determines a point of the depegram. The intersection of the adiabatic through the starting point of the ascending air with the vapor-content line through the corresponding point of the depegram gives the point at which direct adiabatic ascent would bring the air to saturation.

The degree of instability, and the amount of energy available, at any level, for the production of showers, thunderstorms, and other weather disturbances, may be determined as follows: Calculate the weight of water vapor per kilogram of dry air actually present in the air at the level, draw a horizontal line through the corresponding point on the tephigram to the temperature line at which the above weight of water vapor would be the saturation amount; then the available energy of the actual nonsaturated air is represented by the area, if any, inclosed by the saturation adiabatic through the latter point and the tephigram. An investigation by Dines has shown, e. g., that the amount of available energy and the level at which it occurs are intimately connected with the subsequent occurrence of thunderstorms (*5*).

APPENDIX

General considerations concerning the energy in the earths' atmosphere, with references to the literature, are given by Woolard, *Mon. Weath. Rev.*, 54, 254–255, 1926. A lucid exposition of fundamental thermodynamical theory will be found in Birtwistle, *Principles of Thermodynamics* (Cambridge Press, 1925). The thermodynamical equations for the atmosphere are developed by Humphreys, *Physics of the Air* (Philadelphia, 1920); Exner, *Dynamische Meteorologie* (2te aufl., Wien, 1925); Shaw, in *Dict. Appl. Phys.* (Glazebrook, ed., vol. III, London, 1923); and others.

A table of the 0.288th powers of numbers, useful in computing potential temperatures, will be found in the *Quar. Jour. Roy. Met Soc.*, 47, 196–202, 1921. Virtual temperatures are tabulated, and also determined graphically, in Bjerknes, *Dynamic Meteorology and Hydrography*, Pt. I, Washington, 1910. Tables of the saturation pressure and density of water vapor will be found in many places. The weight of water vapor required to saturate a kilogram of dry air is tabulated by Shaw in Table IV of his article in *Dict. Appl. Phys.*, vol. III, and the values of R′ in eq. (7) are given in Table V; total and realized entropy at various pressures and temperatures are set out in Table VI.

A very complete set of equations, tables, and graphs for the dry, rain, hail, and snow stages has been published by J. E. Fjeldstad, *Geofysiske Publik.*, vol. III, No. 13 (Oslo, 1925). The various different forms of thermodynamical diagrams for the atmosphere, with examples of their applications, are given by Shaw, *Air and its Ways* (Cambridge Press, 1923; Figs. 49 and 50), and by Shaw and Fahmy, *Quar. Jour. Roy. Met. Soc.*, 51, 205–228, 1925.

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